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The Nucleon-Nucleon Potential in the $1/N_c$ Expansion

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The nucleon-nucleon potential is analysed using the $1/N_c$ expansion of QCD. The NN potential is shown to have an expansion in $1/N_c^2$, and the strengths of the leading order central, spin-orbit, tensor, and quadratic spin-orbit forces (including isospin dependence) are determined. Comparison with a successful phenomenological potential (Nijmegen) shows that the large- N_c analysis explains many of the qualitative features observed in the nucleon-nucleon interaction. The $1/N_c$ expansion implies an effective Wigner supermultiplet symmetry for light nuclei. Results for baryons containing strange quarks are presented in an appendix.

1. Introduction

The two-body nucleon-nucleon interaction is the basic ingredient that is used in calculating the properties of nuclei. There are a number of phenomenologically successful models for the interaction, typically constructed using meson exchange contributions. Unfortunately there is no direct connection between these models and the underlying theory of the strong interactions, QCD. Until recently, the only way that QCD and the physics of nucleon interactions could be rigorously related has been through symmetry arguments. By making use of symmetry and effective field theory, one can calculate the low energy dynamics using a small number of parameters that are fitted to the experimental data. Such theories can be quite predictive, but still remain somewhat remote from QCD — one would like *a priori* arguments for the sizes of the phenomenological parameters.

Recently, there has been significant progress in better understanding the implications of QCD for hadronic physics by exploiting the “hidden” expansion parameter of QCD — $1/N_c$, where $N_c = 3$ is the number of colors [1-5]. In the large N_c limit, one finds that meson-baryon interactions respect an $SU(4)$ spin-flavor symmetry — the same symmetry found in the nonrelativistic quark model [1,6]. It has also been shown that the $1/N_c$ expansion can provide information about the nucleon-nucleon potential [7]. In particular, ref. [7] analyzed the central potential for NN scattering and showed that the $1/N_c$ expansion gave a qualitative understanding of its spin and isospin structure.

In this paper we extend the analysis of [7] to include the entire NN potential. Naively, one might think that the $N_c \rightarrow \infty$ limit is not relevant for analyzing nuclear physics. Nuclear matter forms a classical crystal at $N_c = \infty$, and so there must be a phase transition between $N_c = 3$ and $N_c = \infty$. While the $1/N_c$ expansion is not reliable for studying bulk properties of nuclear matter, it does allow one to analyze the spin and isospin dependence of the nuclear force. One expects that the symmetry properties of the NN interaction will be independent of the phase of the many body groundstate.

The general form of the potential for elastic, nonrelativistic NN scattering is

$$V_{NN} = V_0^0 + V_\sigma^0 \sigma_1 \cdot \sigma_2 + V_{LS}^0 \mathbf{L} \cdot \mathbf{S} + V_T^0 S_{12} + V_Q^0 Q_{12} \\ + (V_0^1 + V_\sigma^1 \sigma_1 \cdot \sigma_2 + V_{LS}^1 \mathbf{L} \cdot \mathbf{S} + V_T^1 S_{12} + V_Q^1 Q_{12}) \tau_1 \cdot \tau_2 , \quad (1.1)$$

where

$$S_{12} \equiv 3\sigma_1 \cdot \hat{\mathbf{r}} \sigma_2 \cdot \hat{\mathbf{r}} - \sigma_1 \cdot \sigma_2 \\ Q_{12} = \frac{1}{2} \{(\sigma_1 \cdot \mathbf{L}), (\sigma_2 \cdot \mathbf{L})\} . \quad (1.2)$$

TABLE 1

Isospin	V_0	V_σ	V_{LS}	V_T	V_Q
$\mathbf{1} \cdot \mathbf{1}$	N_c	$1/N_c$	$1/N_c$	$1/N_c$	$1/N_c^3$
$\tau_1 \cdot \tau_2$	$1/N_c$	N_c	$1/N_c$	N_c	$1/N_c$

The four terms V_0^i , V_σ^i constitute the central potential, while V_T^i , V_{LS}^i and V_Q^i are the tensor interaction, the spin-orbit interaction, and the quadratic spin-orbit interaction respectively; the ten functions V_a^i can in general be velocity dependent. The main result of this paper is that the strength of the ten functions V_a^i can be determined in the $1/N_c$ expansion, and are as given in Table 1. As is apparent from this Table, the actual expansion parameter is not $1/N_c$ but $1/N_c^2$. Thus even though the actual value $N_c = 3$ is not very large, an expansion in $1/N_c^2$ can be quite predictive.

The organization of this paper is as follows: in §2 we briefly review general properties of baryons in the large N_c limit. In §3 we derive the results given in Table 1. These results are compared in §4 with the “Nijmegen potential” of references [8,9] — a phenomenologically successful model of the NN interaction; we show that the hierarchy of Table 1 is evident in NN phenomenology. §5 extends the discussion of ref. [7] concerning how the Wigner supermultiplet symmetry might arise in light nuclei as a consequence of the $1/N_c$ expansion. This is followed by conclusions, and an appendix in which we extend our analysis to hyperon interactions (*i.e.*, including the s quark).

2. The Large N_c QCD Analysis

The large N_c limit is defined by taking the number of colors N_c of QCD to be large while simultaneously rescaling the QCD coupling as $g \rightarrow g/\sqrt{N_c}$, keeping Λ_{QCD} fixed [10]. The $1/N_c$ expansion has proven to be a powerful tool for analyzing baryon properties, since baryon structure simplifies considerably in this limit. Antiquarks in the baryon are suppressed, and as baryons consist of N_c quarks interacting with $1/N_c$ strength, the Hartree approximation becomes exact in the large N_c limit [11]. Although one cannot solve the Hartree equations due to the nonlinearity of glue interactions, one can nevertheless determine a number of useful properties of the spin and flavor properties of the baryons and their interactions.

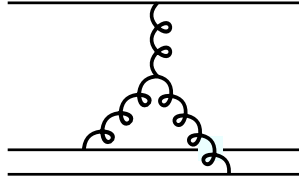


Fig. 1. A QCD contribution to H leading in $1/N_c$. This diagram can be described in spin-flavor space as a product of three of the 1-quark operators given in eq. (2.1).

To analyze the flavor and spin structure of the Hartree Hamiltonian in the case of two light flavors, it is convenient to use as an operator basis the one-quark operators of the quark model

$$\hat{S}^i = q^\dagger \frac{\sigma^i}{2} q, \quad \hat{I}^a = q^\dagger \frac{\tau^a}{2} q, \quad \hat{G}^{ia} = q^\dagger \frac{\sigma^i \tau^a}{4} q, \quad (2.1)$$

where $q = (u, d)$ and q^\dagger are the creation and annihilation operators for the u and d quark flavors, and σ^i, τ^a are the standard $SU(2)$ Pauli matrices acting on spin and isospin respectively. The q and q^\dagger operators do not carry color, and are bosonic. The Hartree Hamiltonian can then be constructed as monomials of these operators. An important result from large N_c QCD is that the Hartree Hamiltonian takes the form [3-5]:

$$H = N_c \sum_n \sum_{s,t} v_{stn} \left(\frac{\hat{S}}{N_c} \right)^s \left(\frac{\hat{I}}{N_c} \right)^t \left(\frac{\hat{G}}{N_c} \right)^{n-s-t} \quad (2.2)$$

where the operators $\{\hat{S}, \hat{I}, \hat{G}\}$ are given in eq. (2.1), the coefficients v are $\mathcal{O}(1)$ functions of momenta, and we have suppressed isospin, spin, and vector indices which are contracted such that H is rotation and isospin invariant. An example of a contribution to H is pictured in fig. 1. It is important that although we make use of the quark model operator basis, eq. (2.2) makes no assumption about the validity of the quark model; the quark model operators are a representation of the spin-flavor Clebsch-Gordon coefficients, and provide an efficient way of doing group theory computations. A Skyrme model basis, for example, would have worked just as well [12].

The lowest lying eigenstates of H with baryon number $B = 1$ have isospin I and spin S satisfying $I = S = \frac{1}{2}, \frac{3}{2}, \frac{5}{2} \dots$. The first two states can be identified with the N and Δ ; the higher spin states do not exist for $N_c = 3$. To leading order, this baryon tower is degenerate with mass $M \sim N_c$. One can show that it transforms as an irreducible representation (the

totally symmetric N_c -index tensor) of an approximate $SU(4)$ spin-flavor symmetry; this is the symmetry under which the quark operators $u \uparrow$, $u \downarrow$, $d \uparrow$, $d \downarrow$ transform as the four-dimensional fundamental representation [1,13]. For $N_c = 3$ the $\{N, \Delta\}$ spin states transform as the 20 dimensional representation of $SU(4)$, familiar from the quark model.

Additional information can be obtained by considering matrix elements of operators between baryon states B and B' restricted to have $S = I \sim 1$. For example, matrix elements of the basis operators (2.1) satisfy

$$\langle B' | \hat{S}/N_c | B \rangle \sim \langle B' | \hat{I}/N_c | B \rangle \sim 1/N_c , \quad \langle B' | \hat{G}/N_c | B \rangle \sim 1 . \quad (2.3)$$

Matrix elements of many body operators can be analyzed as well, using various relations among powers of the basic operators \hat{S}^i , \hat{I}^a , and \hat{G}^{ia} [1,3]. This allows one to greatly reduce the number of linearly independent terms in (2.2) at a given order in $1/N_c$. Using these techniques, one is able to show that the matrix element of a general n -quark operator $\hat{\mathcal{O}}_{I,S}^{(n)}$ with $B = 0$, isospin I and spin S , is of size [3] (see also [7] for a derivation)

$$\langle B' | \hat{\mathcal{O}}_{I,S}^{(n)}/N_c^n | B \rangle \lesssim 1/N_c^{|I-S|} . \quad (2.4)$$

The fact that the operators with the largest matrix elements have $I = S$ was first observed in the Skyrme model [14], and is known as the “ $I_t = J_t$ rule”.

Eqs. (2.2)–(2.4) are the central results behind the large N_c analysis of baryons. One consequence is that the mass splittings in the baryon tower (e.g, between N and Δ) are of size $1/N_c$ [2,11]. This result is in good agreement with the real world where the ratio $R \equiv (M_\Delta - M_N)/(M_\Delta + M_N) = 0.13$, while the large N_c prediction at $N_c = 3$ is $R \sim 1/N_c^2 = 0.11$. Consequences of eq. (2.3) for the NN interaction are explored in the next section.

3. The Nucleon-Nucleon Interaction

There are two independent three-momenta for baryon-baryon scattering in the center of mass frame, which can conveniently taken to be

$$\mathbf{q} = \mathbf{p}_{in} - \mathbf{p}_{out} , \quad \mathbf{k} = \mathbf{p}_{in} + \mathbf{p}_{out} . \quad (3.1)$$

These momenta are to be considered independent of N_c in the $1/N_c$ expansion. To leading order in $1/N_c$, the entire baryon tower is degenerate and $|\mathbf{p}_{in}| = |\mathbf{p}_{out}|$ for elastic scattering,

up to $\mathcal{O}(1/N_c^2)$ corrections, and so $\mathbf{q} \cdot \mathbf{k} = 0$ to the same order. The general baryon-baryon interaction potential is then a matrix

$$V(\mathbf{q}, \mathbf{k}) = \langle \mathbf{p}_{out}, \gamma; -\mathbf{p}_{out}, \delta | H | \mathbf{p}_{in}, \alpha; -\mathbf{p}_{in}, \beta \rangle \quad (3.2)$$

where H is the Hartree Hamiltonian (2.2) and α, \dots, δ denote internal quantum numbers of the baryons, such as spin, flavor and particle type (*e.g.* N or Δ). Throughout this paper we will define the NN potential as the above matrix element restricted to the space of nucleons. We do not consider second order effects due to virtual Δ 's, etc.

There are two ways that $1/N_c$ factors can suppress terms in the potential. The first arises from spin-flavor structure and the powers of $1/N_c$ in eq. (2.3). The second source of suppression arises in velocity dependent interactions arising as relativistic corrections. Since the nucleon velocity equals $\mathbf{p}/M \sim 1/N_c$, each power of velocity is equivalent to a $1/N_c$ suppression. In the nonrelativistic limit for baryons, a t -channel meson exchange contribution to V is only a function of \mathbf{q} . A u -channel contribution is only a function of \mathbf{k} , and can be expressed as an exchange potential. Relativistic corrections allow a single meson exchange contribution to V to be a function of both \mathbf{q} and \mathbf{k} . Meson exchange in the t -channel is then a function of \mathbf{q} and \mathbf{k}/M , with each power of \mathbf{k} being accompanied by one factor of M . Similarly, u -channel meson exchange is a function of \mathbf{k} and \mathbf{q}/M . This shows that if a general velocity dependent potential is expanded in a Taylor series in \mathbf{k} and \mathbf{q} , a term of the form $\mathbf{q}^r \mathbf{k}^s$ is suppressed by

$$1/N_c^n, \quad n = \text{Min}(r, s). \quad (3.3)$$

Combining this source of $1/N_c$ suppression with eq. (2.3) will allow us to determine the size and spin-flavor structure of the dominant terms in the potential V .

An NN interaction at the QCD level gets contributions from complicated processes, such as pictured in fig. 2. Each of these contributions can be expressed as a tensor function $v(\mathbf{q}, \mathbf{k})$ contracted with 1-quark operators \hat{S}^i , \hat{I}^a and \hat{G}^{ia} which act on either of the two nucleon states. The coefficient function v_{stn} in eq. (2.2) and the operators \hat{S}^i and \hat{G}^{ia} must combine to be invariant under rotations. Our analysis is simplified by first expanding v_{stn} (and hence V) in multipole moments; the following subsections are organized accordingly as $\Delta L = 0$ (the central force), $\Delta L = 1$ (spin-orbit force), and $\Delta L = 2$ (the tensor and quadratic spin-orbit forces).

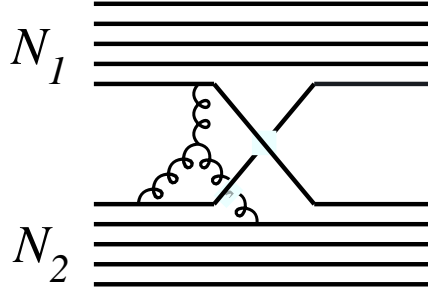


Fig. 2. An example of a contribution to the NN interaction from the 3-quark operator pictured in fig. 1. This diagram can be described in spin-flavor space as a single 1-quark operator acting on the first baryon N_1 , and two 1-quark operators acting on the N_2 line.

3.1. $\Delta L = 0$: The Central Potential

This is the case analyzed in ref. [7]. The central force can be written as a sum of products of 1-quark operators as in eq. (2.2), where the operators act on either the N_1 or N_2 nucleon states, and the coefficients v_{stn} are general scalar functions of $|\mathbf{q}|$ and $|\mathbf{k}|$. It follows from eq. (2.3) that the leading contribution will have no \hat{S}^i/N_c or \hat{I}^a/N_c operators, since each of these implies a $1/N_c$ suppression; instead it will consist solely of powers of \hat{G}^{ia}/N_c . By rotational symmetry, since the v coefficients are scalars, the \hat{G}^{ia} operators must be contracted to form spin invariants. Similarly, isospin symmetry implies that the \hat{G}^{ia} must be contracted to form isospin invariants. From these constraints, it is possible to show that at leading order in $1/N_c$, the most general form for the central potential is [7]

$$V_{\text{central}} = N_c \sum_{n=0}^{N_c} v_n \left(\frac{\hat{G}_1 \cdot \hat{G}_2}{N_c^2} \right)^n, \quad (3.4)$$

where $\hat{G}_1 \cdot \hat{G}_2 \equiv \hat{G}_1^{ia} \hat{G}_2^{ia}$. In general, the coefficients v_n are functions of both $|\mathbf{q}|^2$, $|\mathbf{k}|^2$, and obey the rule eq. (3.3). One can further restrict the powers of $\hat{G}_1 \cdot \hat{G}_2$ in eq. (3.4) to be completely symmetric in the \hat{G}_1 indices, and in the \hat{G}_2 indices, before the two sets of indices are contracted.

It is straightforward to verify that (3.4) is the most general form of the leading order $\Delta L = 0$ potential. We have argued that it can only involve powers of the \hat{G}^{ia} operators, on the basis of eq. (2.3); what must be shown is that the indices are contracted as above in eq. (3.4). By the operator reduction rule [3] any terms in which two indices of $\hat{G}^{ia} \hat{G}^{jb}$ (where both \hat{G} 's act on the same baryon) are contracted with each other by δ^{ij} , δ^{ab} , ϵ^{ijk}

or ϵ^{abc} can be eliminated in favor of terms with fewer powers of \hat{G} . Thus the only allowed invariants are obtained by contracting the indices of \hat{G}_1^{ia} with those of \hat{G}_2^{ia} , as in (3.4). More complicated contractions, such as

$$\hat{G}_1^{ia} \hat{G}_1^{jb} \hat{G}_2^{ib} \hat{G}_2^{ja} \quad (3.5)$$

can be written as

$$\begin{aligned} & \left(\hat{G}_1 \cdot \hat{G}_2 \right)^2 + \hat{G}_1^{ia} \hat{G}_1^{jb} \left[\hat{G}_2^{ib} \hat{G}_2^{ja} - \hat{G}_2^{ia} \hat{G}_2^{jb} \right] \\ &= \left(\hat{G}_1 \cdot \hat{G}_2 \right)^2 - \epsilon^{abc} \hat{G}_1^{ia} \hat{G}_1^{jb} \epsilon^{ghc} \hat{G}_2^{ig} \hat{G}_2^{jh}. \end{aligned} \quad (3.6)$$

The term with two ϵ symbols can be reduced to $\hat{G}_1 \cdot \hat{G}_2$ using the relations in [3], so that all contractions of \hat{G}_1 with \hat{G}_2 can be written as powers of $\hat{G}_1 \cdot \hat{G}_2$. One can also restrict the indices on powers of \hat{G}_1 and \hat{G}_2 to be completely symmetrized, since terms antisymmetric in the indices can be eliminated using the operator identities. The series in $\hat{G}_1 \cdot \hat{G}_2$ terminates after N_c terms, because an operator with more than N_c quark fields acting on a single baryon can be reexpressed in terms of operators with $\leq N_c$ quark fields.¹

There are no $1/N_c$ corrections to (3.4), except through $1/N_c$ dependence in the unknown coefficients v_n . All terms in the $1/N_c$ correction to (3.4) are arbitrary polynomials in $G_{1,2}^{ia}$, with one factor of $S_{1,2}^i$ or $I_{1,2}^a$. It is easy to check that all such terms have the wrong time-reversal properties to contribute to the baryon-baryon potential. Thus the first correction to (3.4) contains a factor of $\hat{S}_1 \cdot \hat{S}_2$ or $\hat{I}_1 \cdot \hat{I}_2$ and is of order $1/N_c^2$.

Eqs. (2.2), (2.4), and (1.1) define the N_c -counting for the central potential. Large N_c QCD implies that the central potential is of order N_c , but is determined by only two independent functions instead of four (at leading order in $1/N_c$):

$$V_0^0(r) \sim N_c, \quad V_\sigma^1(r) \sim N_c, \quad (3.7)$$

while

$$V_\sigma^0(r) \sim 1/N_c, \quad V_0^1(r) \sim 1/N_c. \quad (3.8)$$

As was noted in [7] and will be discussed in §5, the above relation implies that the central potential obeys an effective Wigner supermultiplet symmetry.

¹ An easy way to see this is to normal order the operators. A normal ordered product involving with more than N_c quark operators on a baryon vanishes, which gives the desired identity.

3.2. $\Delta L = 1$: The Spin-Orbit Potential

The $\Delta L = 1$ baryon interaction amplitude contains the spin-orbit coupling term; it is obtained from the general Hartree Hamiltonian eq. (2.2) by restricting attention to terms for which the coefficient v transforms as a vector under rotations. It follows that the 1-quark operators multiplying the v coefficients must be combined to transform as a $(1,0)$ representation under $SU(2)_{spin} \times SU(2)_{isospin}$. From eq. (2.3) we have seen that to contribute at leading order in $1/N_c$, an n -quark operator must be a polynomial in the \hat{G} 's alone. However, one cannot make a $(1,0)$ operator with the correct parity and time reversal properties purely out of \hat{G} 's. The spin-orbit force is suppressed relative to the central force, and is an arbitrary polynomial in \hat{G} 's, with one factor of \hat{S} or \hat{I} . The general form of the $\Delta L = 1$ amplitude is

$$\begin{aligned}
V_{LS} = & N_c \sum_{n=0}^{N_c-1} v_{1,n}^i \left(\frac{\hat{S}_1^i + \hat{S}_2^i}{N_c} \right) \left(\frac{\hat{G}_1 \cdot \hat{G}_2}{N_c^2} \right)^n \\
& + N_c \sum_{n=0}^{N_c-2} v_{2,n}^i \left(\frac{\hat{G}_2^{ia} \hat{I}_1^a + \hat{G}_1^{ia} \hat{I}_2^a}{N_c^2} \right) \left(\frac{\hat{G}_1 \cdot \hat{G}_2}{N_c^2} \right)^n \\
& + N_c \sum_{n=0}^{N_c-3} v_{3,n}^i \left(\frac{\hat{G}_1^{ia} \hat{G}_2^{ja} \hat{S}_1^j + \hat{G}_2^{ia} \hat{G}_1^{ja} \hat{S}_2^j}{N_c^3} \right) \left(\frac{\hat{G}_1 \cdot \hat{G}_2}{N_c^2} \right)^n.
\end{aligned} \tag{3.9}$$

This can be derived by arguments similar to those in the previous subsection. Time reversal and parity invariance requires the coefficients $v_{m,n}^i$ in (3.9) to be proportional to $(\mathbf{q} \times \mathbf{k})$ times an arbitrary function of \mathbf{q}^2 and \mathbf{k}^2 . In position space, a contribution of the form $U(\mathbf{q}^2) (\mathbf{q} \times \mathbf{k}) \cdot (\mathbf{S}_1 + \mathbf{S}_2)$ is of the form $(\nabla U(r) \times \mathbf{k}) \cdot \mathbf{S}$, which is the usual spin-orbit force. There is a hidden suppression factor of $1/N_c$ (which follows from eq. (3.3)) in the spin-orbit force which is not manifest in eq. (3.9), since the $\Delta L = 1$ interaction necessarily involves both \mathbf{q} and \mathbf{k} .

The Wigner-Eckart theorem implies that there are only two distinct operators when the expression (3.9) for the $\Delta L = 1$ amplitude is restricted to the nucleon sector. These are the two spin-orbit terms appearing in eq. (1.1). Thus we find

$$V_{LS}^0(r) \sim 1/N_c, \quad V_{LS}^1(r) \sim 1/N_c. \tag{3.10}$$

The spin-orbit force is $\mathcal{O}(1/N_c^2)$ in strength relative to the central force, and it is of comparable strength in the two isospin channels.

3.3. $\Delta L = 2$: The Tensor and Quadratic Spin-Orbit Potentials

The $\Delta L = 2$ amplitude is obtained by requiring that the coefficients v in eq. (2.2) transform under rotations as $\Delta L = 2$. The leading order amplitude is a polynomial in the \hat{G} 's that transforms as $S = 2$, $I = 0$. One can obtain an amplitude that does not violate the $I_t = J_t$ rule on each baryon line by combining $I = S = 1$ amplitudes on each baryon into total $I = 0$ and total $\Delta L = 2$. The general form of the leading order amplitude is

$$V_T^1 = N_c \sum_{n=0}^{N_c-1} v_n^{ij} \frac{\hat{G}_1^{ia} \hat{G}_2^{ja}}{N_c^2} \left(\frac{\hat{G}_1 \cdot \hat{G}_2}{N_c^2} \right)^n \quad (3.11)$$

where the coefficient v_n^{ij} is a symmetric traceless tensor that depends on \mathbf{q} and \mathbf{k} . Time reversal invariance requires the coefficients to have the form

$$v_n \times \left(\mathbf{q}^i \mathbf{q}^j - \frac{1}{3} \mathbf{q}^2 \delta^{ij} \quad \text{or} \quad \mathbf{k}^i \mathbf{k}^j - \frac{1}{3} \mathbf{k}^2 \delta^{ij} \right),$$

where v_n is a scalar function of \mathbf{q}^2 , \mathbf{k}^2 and $(\mathbf{q} \cdot \mathbf{k})^2$.

If one restricts the interaction eq. (3.11) to the nucleon sector, one gets

$$V_T^1 = N_c v_n \tau_1 \cdot \tau_2 \left(\mathbf{q} \cdot \boldsymbol{\sigma}_1 \mathbf{q} \cdot \boldsymbol{\sigma}_2 - \frac{1}{3} \mathbf{q}^2 \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 \right). \quad (3.12)$$

Terms with $n > 1$ in eq. (3.11) can be dropped, because two spin-1/2 nucleons can only give non-zero matrix elements for operators with spin ≤ 1 . Comparing with eq. (1.1), we see that

$$V_T^1 \sim N_c. \quad (3.13)$$

The other term in the tensor potential, V_T^0 , has $|I - S| = 1$ at each nucleon line, and so by eq. (2.4)

$$V_T \sim 1/N_c. \quad (3.14)$$

A similar and straightforward analysis for V_Q gives the results listed in Table 1.

3.4. The NN potential and the Δ

One flaw in our discussion that should be eventually improved upon is the treatment of the Δ . The Hartree Hamiltonian (2.2) implicitly acts on the entire $I = S$ baryon tower, including both nucleons and Δ 's, all of which are degenerate in the $N_c \rightarrow \infty$ limit. In our discussion of the NN potential, we have simply projected H to the nucleon sector. A more sophisticated treatment would be to integrate the Δ 's out of the theory (keeping track of the $1/N_c$ mass splitting) and to construct an effective theory for nucleons alone. This is a subtle analysis (see, for example, [15]) and beyond the scope of this paper.

4. Comparison of large N_c QCD with a phenomenologically successful model

Our large- N_c results for the general nucleon-nucleon potential of eq. (1.1) are displayed in Table 1. For two flavors we have found that the strongest NN interactions are the central force terms V_0 and V_σ^τ , as well as the tensor force V_T^τ , all three of which are $\sim N_c$. The remaining contributions to the NN potential, with the exception of V_Q , are relatively suppressed by $\sim \mathcal{O}(1/N_c^2)$. Finally, the isospin invariant quadratic spin-orbit force V_Q is suppressed by $\sim \mathcal{O}(1/N_c^4)$ compared to the central potential, as it is both an $I \neq S$ interaction, as well as being a second order relativistic effect suppressed by $1/M^2$. The results we have derived are consistent with the $I_t = J_t$ rule, but are more general. They are true in QCD in the $1/N_c$ expansion, and make no assumptions about the origin of the NN interaction as being, for example, due to one meson exchange.

The results can be directly compared with nuclear potential models in momentum space. A particularly simple phenomenological model to compare with is the meson exchange model “Nijmegen potential” of references [8,9]. In this model, the NN potential is approximated in momentum space by a sum of Yukawa and Gaussian interactions times powers of momenta divided by masses, contracted with the spin and isospin Pauli matrices. The Yukawa potentials correspond to one-particle exchange of both real mesons ($\pi, \eta, \eta', \rho, \omega, \phi, a_0, f_0$) and an “effective meson” (ϵ), while the Gaussian potentials are labelled by P, f_2, f'_2 and a_2 . The motivation for this form of the potential is unimportant here; it provides a phenomenologically successful parametrization for the NN potential that can be compared with Table 1. The N_c dependence should appear in the relative strengths of the potentials, and the $1/M$ factors that appear when the potential is decomposed as in eq. (1.1). The strength of the contributions to the Nijmegen potential are simple to evaluate, since they are presented explicitly in momentum space, and we can treat all momenta and meson masses as ~ 1 in the $1/N_c$ expansion. The N_c dependence must then reside in the strengths of the couplings used in the Nijmegen potential, as well as the explicit factors of the nucleon mass that appear in the formulas of ref. [8]. One finds for the strength of the various terms in the potential

$$\begin{aligned}
V_0^I &\sim g_{I0}^2, \frac{g_{I0}g_{I1}\Lambda}{M}, \frac{g_{I1}^2\Lambda^2}{M^2}, \\
V_\sigma^I &\sim V_T^I \sim \frac{g_{I0}^2}{M^2}, \frac{g_{I0}g_{I1}}{\Lambda M}, \frac{g_{I1}^2}{\Lambda^2}, \\
V_{LS}^I &\sim \frac{g_{I0}^2}{M^2}, \frac{g_{I0}g_{I1}}{\Lambda M}, \\
V_Q^I &\sim \frac{g_{I0}^2}{M^4}, \frac{g_{I0}g_{I1}}{\Lambda M^3}, \frac{g_{I1}^2}{\Lambda^2 M^2},
\end{aligned} \tag{4.1}$$

where $I = 0, 1$ correspond to the $1 \cdot 1$ and $\tau_1 \cdot \tau_2$ isospin structures respectively, M is the nucleon mass, and Λ is a strong interaction scale characterizing the derivative expansion (denoted \mathcal{M} in [8]). The parameters g_{IS} correspond to the coupling constants of the model with t -channel (isospin, spin) = (I, S) in the nonrelativistic limit; in particular, the scalar coupling g_S and vector couplings g_V and f_V of ref. [8] are given by g_{I0} , g_{I0} and g_{I1} respectively, where I is the meson isospin. (The pseudoscalar contributions are parametrized differently in [8] and are mentioned below). As far as the N_c scaling goes, $M \sim N_c$, while the Λ and the masses of the exchange mesons are all ~ 1 . In eq. (4.1) we have omitted dimensionful quantities that do not scale with N_c , such as the meson propagators $1/(q^2 + m^2)$. By comparing the expressions in eq. (4.1) with our results in Table 1, one sees that they are consistent provided that the couplings g_{IS} scale with N_c as

$$g_{IS} \propto N_c^{(1/2 - |I-S|)} . \quad (4.2)$$

This N_c scaling can be compared with the numerical values given in ref. [9]. In fig. 3 we have plotted the couplings determined numerically in ref. [9], rescaled by their value for f_ρ . Since f_ρ is a g_{11} coupling, eq. (4.2) implies that the leading large- N_c prediction for the ratio is

$$\hat{g}_{IS} \equiv \frac{g_{IS}}{f_\rho} = \begin{cases} 1, & \text{if } |I - S| = 0 \\ \frac{1}{3}, & \text{if } |I - S| = 1 \end{cases} . \quad (4.3)$$

As can be seen from fig. 3, there is good qualitative agreement between the large- N_c prediction (4.3), and the g_{IS} values used in the Nijmegen potential. Omitted from fig. 3 are the pseudoscalar couplings, which are not readily compared with heavy meson couplings, due to their special status as pseudo-Goldstone bosons. However, the pseudoscalar meson couplings are related to the axial current couplings, which have been analyzed in detail, and shown to agree with $1/N_c$ predictions [16]. There are two couplings in the Nijmegen potential, the ϕ and a_2 coupling, that deviate significantly from the $1/N_c$ pattern. The ϕ meson is a pure $\bar{s}s$ state, and only couples to the nucleons through quark loops. Its coupling is OZI suppressed, and should be of order $1/N_c$ relative to the ω couplings. The Nijmegen fit has $g_\phi/g_\omega \approx 0.1$, which is a factor of three smaller than the naive $1/N_c$ prediction. The a_2 coupling is even somewhat smaller.

It must be stressed that the numerical parameters plotted in fig. 3 were obtained by treating the couplings as phenomenological parameters in the NN potential, chosen to provide the best fit to NN scattering data. There is no reason to assume that the NN force is actually due to single meson exchange; in fact, the ϵ and P contributions to the

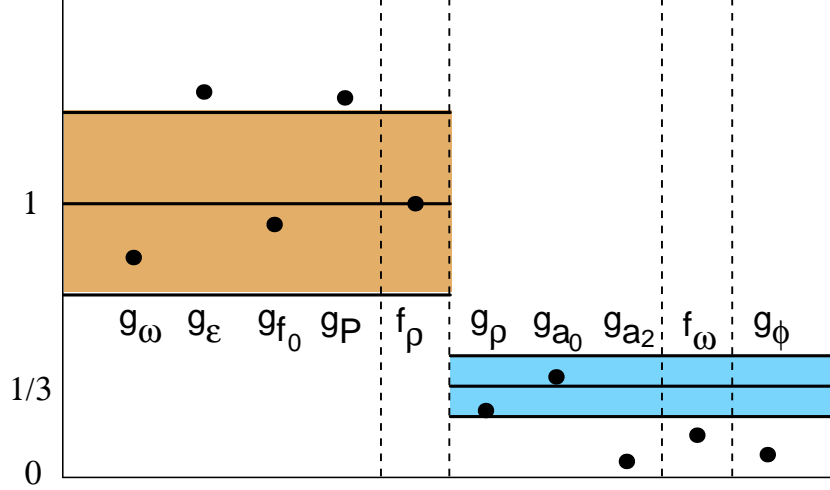


Fig. 3. The couplings for the Nijmegen NN potential in ref. [9] rescaled by f_ρ . The values for this ratio predicted by large N_c QCD in eq. (4.3) are indicated by lines, and the shaded regions are the size of the expected $\mathcal{O}(1/N_c)$ corrections to the leading result. The five regions in the plot (separated by vertical dashed lines) are (from left to right) the $(I, S) = (0, 0)$, $(1, 1)$, $(1, 0)$, $(0, 1)$, and g_ϕ couplings.

potential do not correspond to single meson exchange at all, and the a_2 in the Nijmegen potential has a Gaussian propagator. The model subsumes such effects as $2\text{-}\pi$ exchange, $\rho\pi$ exchange, etc. within the phenomenological couplings g_{IS} . Only the pseudoscalar meson couplings are related to the physical meson-nucleon couplings, since the long distance part of the NN potential is dominated by single meson exchange. Thus the agreement between fig. 3 and the large- N_c prediction (4.3) contains more than the claim that meson-baryon couplings obey the $I_t = J_t$ rule. We take fig. 3 to provide encouraging evidence that our large- N_c analysis of the NN interaction describes the qualitative features seen in nature.

5. The Central Potential and Wigner Supermultiplet Symmetry

It was suggested in ref. [7] that the approximate Wigner supermultiplet symmetry observed in light nuclei could be explained by the $1/N_c$ expansion of QCD. Under the Wigner symmetry $SU(4)_W$, the four *nucleon* states $p \uparrow$, $p \downarrow$, $n \uparrow$ and $n \downarrow$ transform as the four dimensional fundamental representation. Note that $SU(4)_W$ is distinct from the quark model $SU(4)$, and that the former cannot be realized as a symmetry at the quark level. Nevertheless, ref. [7] argued that the $1/N_c$ expansion explains how $SU(4)_W$ symmetry could emerge as an accidental symmetry in light nuclei. As that work only examined the central part of the NN potential, it is worth reexamining the argument.

Under $SU(4)_W$ symmetry, a two-nucleon state transforms like $\mathbf{4} \times \mathbf{4} = \mathbf{6}_A + \mathbf{10}_S$, where the subscripts A and S denote the antisymmetric and symmetric combinations. Under spin \times isospin, these representations decompose as

$$\begin{aligned}\mathbf{6} &\rightarrow (0, 1) + (1, 0), \\ \mathbf{10} &\rightarrow (0, 0) + (1, 1) .\end{aligned}\tag{5.1}$$

If the two nucleons are in an even partial wave, they must be in a totally antisymmetric spin \otimes isospin state, so they are in a $(0, 1)$ or $(1, 0)$ state, i.e. in $\mathbf{6}_A$ of $SU(4)_W$. If the two nucleons are in an odd partial wave, they must be in a totally symmetric spin \otimes isospin state, so they are in a $(0, 0)$ or $(1, 1)$ state, i.e. in $\mathbf{10}_S$ of $SU(4)_W$.

We have shown in §3 that the leading contributions to the NN potential (at strength N_c) are V_0^0 , V_σ^1 and V_T^1 . The first two terms correspond to the operators

$$1, \quad \sigma_1 \cdot \sigma_2 \quad \tau_1 \cdot \tau_2,\tag{5.2}$$

which have the same value on $(0, 1)$ or $(1, 0)$, i.e. they have the same value on the entire $\mathbf{6}$ representation of $SU(4)_W$. Thus at leading order in N_c , the central potential respects Wigner $SU(4)_W$ symmetry if the two nucleons are in an even partial wave. Violation of Wigner $SU(4)_W$ from the central potential in the even partial waves is an $\mathcal{O}(1/N_c^2)$ effect. The operators eq. (5.2) have different values on the $(0, 0)$ and $(1, 1)$ representations, and so break $SU(4)_W$ symmetry when acting on the $\mathbf{10}$ representation of $SU(4)_W$. Thus the central potential breaks Wigner $SU(4)_W$ symmetry at leading order in the odd partial waves. The tensor force V_T^1 also violates Wigner $SU(4)_W$ symmetry at leading order, in all partial waves.

Nevertheless, there is reason to expect to see Wigner symmetry in light nuclei. The nucleons inside a nucleus have low momentum, so the dominant interaction is s -wave scattering, with higher partial waves being kinematically suppressed. Furthermore, the tensor mean field is small in nuclei. Therefore all of the leading order violations of $SU(4)_W$ may be expected to be small.

So why is $SU(4)_W$ not evident in heavy nuclei? At subleading order (a relative $1/N_c^2$), the potentials V_σ^0 , V_0^1 , V_T^0 and $V_{LS}^{0,1}$ all break the Wigner symmetry. The mean fields of all but the spin-orbit force are small in nuclei. However, the importance of the spin-orbit force grows like $A^{1/3}$, proportional to the number of particles in the maximum angular momentum shell. Therefore, for large A , the spin-orbit force is expected to overcome the $1/N_c^2$ suppression and destroy the approximate Wigner supermultiplet symmetry. It may be interesting to pursue this further, to determine at what values of A one might expect $SU(4)_W$ symmetry to fail.

6. Summary and Conclusions

The $1/N_c$ expansion has been shown elsewhere to be a useful tool in analyzing the properties of baryons ²; the analysis presented here and in ref. [7] shows that it also provides a useful tool for understanding qualitative features of the nuclear force. In particular, we have computed the relative strengths of the various components of the NN interaction in the $1/N_c$ expansion (Table 1) and argued that the predicted patterns are reproduced in phenomenological models of the NN force (fig. 3). We also extended the argument of ref. [7] that the approximate Wigner supermultiplet symmetry observed in light nuclei (see [7] for examples and references) is in fact understandable in terms of the $1/N_c$ expansion. We are aware of no other explanations for this peculiar $SU(4)_W$ symmetry.

Aside from obtaining directly from QCD a qualitative explanation for the spin, isospin and tensor structure of the NN potential, it is hoped that the $1/N_c$ expansion could serve as a guide toward better understanding the interactions of baryons with strangeness, where the experimental data is much poorer. To this end we have included the three flavor analysis in Appendix A. Our hope is that this could prove useful for understanding hypernuclei, as well as matter in extreme conditions where strangeness may play a significant role, such as in heavy ion collisions, or dense matter with kaon condensation [19] or hyperons.

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² See for example, [17,18].

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Appendix A. Three Flavors

The quark operator basis for three flavors is denoted by

$$\hat{S}^i = Q^\dagger \frac{\sigma^i}{2} Q, \quad \hat{T}^A = Q^\dagger T^A Q, \quad \hat{G}^{iA} = Q^\dagger \frac{\sigma^i}{2} T^A Q, \quad (\text{A.1})$$

where $Q = (u, d, s)$ and Q^\dagger are the creation and annihilation operators for the three light quark flavors, $i = 1, 2, 3$ and $A = 1, \dots, 8$. T^a are the standard $SU(3)$ matrices in the fundamental representation, normalized so that $\text{tr} T^A T^B = \delta^{AB}/2$. These 1-quark operators act on a baryon state which is the completely symmetric tensor product (in spin \otimes flavor) of N_c quarks.

It is convenient to break the operator basis (A.1) for the $SU(6)$ generators by separating $Q = (u, d, s)$ into $q = (u, d)$ and s . Under this decomposition \hat{S}^i , \hat{T}^A and \hat{G}^{iA} break up into linear combinations of

$$\begin{aligned} \hat{S}^i &= q^\dagger \frac{\sigma^i}{2} q, & \hat{I}^a &= q^\dagger \frac{\tau^a}{2} q, & \hat{G}^{ia} &= q^\dagger \frac{\sigma^i \tau^a}{4} q, \\ \hat{S}_s^i &= s^\dagger \frac{\sigma^i}{2} s, & \hat{N}_s &= s^\dagger s, & \hat{Y}^{i\alpha} &= s^\dagger \frac{\sigma^i}{2} q^\alpha, & \hat{K}^\alpha &= s^\dagger q^\alpha, \end{aligned} \quad (\text{A.2})$$

and $\hat{Y}^{i\alpha\dagger}$ and $\hat{K}^{\alpha\dagger}$ which are the hermitian conjugates of $\hat{Y}^{i\alpha}$ and \hat{K}^α . For baryons with N_c quarks, and strangeness of order one, \hat{G}^{iA} is of order N_c , $\hat{Y}^{i\alpha}$ and \hat{K}^α are of order $\sqrt{N_c}$, and \hat{S}_{ud}^i , \hat{I}^a , \hat{S}_s^i and \hat{N}_s are of order one [3]. Note that $\hat{Y}^{i\alpha}$ and \hat{K}^α are strangeness changing operators.

The (properly normalized) $SU(6)$ generators $\sqrt{2}\hat{G}^{iA}$, $\hat{T}^A/\sqrt{2}$ and $\hat{S}^i/\sqrt{3}$ are collectively denoted by $\hat{\Lambda}^M$. The operator basis for two and three flavors are summarized in Table 2. An expansion using the operator basis (A.2) gives us the predictions of the $1/N_c$ expansion for three flavors, without assuming $SU(3)$ symmetry. One can also impose $SU(3)$ symmetry, which places additional restrictions on the final result. The results for two flavors are obtained by using only the operators \hat{S}^i , \hat{I}^a , and \hat{G}^{iA} .

TABLE 2

# of Flavors		Spin	Flavor	Spin – Flavor	All
2		\hat{S}^i	\hat{I}^a	\hat{G}^{iA}	$\hat{\lambda}^\mu$
3		\hat{S}^i	\hat{T}^A	\hat{G}^{iA}	$\hat{\Lambda}^M$
3 \rightarrow 2	$\Delta S = 0$ $\Delta S = 1$ $\Delta S = -1$	\hat{S}^i	\hat{I}^a, \hat{N}_s \hat{K}^α \hat{K}_α^\dagger	$\hat{G}^{iA}, \hat{S}_s^i$ $\hat{Y}^{i\alpha}$ $\hat{Y}_\alpha^{i\dagger}$	

The results of the paper can be generalized to the case of three light flavors. The analysis is more complicated because one also has to include operators \hat{N}_s , \hat{S}_s^i , $\hat{Y}^{i\alpha}$ and \hat{t}^α that involve the s quark. We will simply give the final results here.

The $1/N_c$ $\Delta L = 0$ interaction is

$$\frac{\mathcal{A}_1^{j=0}}{N_c} = \sum_{r=0}^{N_c} c_{1,r} \left(\frac{\hat{\Lambda}_1 \cdot \hat{\Lambda}_2}{N_c^2} \right)^r + \sum_{r=0}^{N_c-1} c_{2,r} \epsilon \frac{\hat{N}_{s1} + \hat{N}_{s2}}{N_c} \left(\frac{\hat{\Lambda}_1 \cdot \hat{\Lambda}_2}{N_c^2} \right)^r. \quad (\text{A.3})$$

The N_s term violates $SU(3)$ symmetry, so its coefficient is proportional to $SU(3)$ breaking in the baryon sector, which is parameterized by ϵ , a dimensionless number of order 0.3. It is clear from Eq. (A.3) that the N_s term violates $SU(6)$ symmetry but respects $SU(4)$ symmetry, so that $SU(6)$ violation is of order ϵ/N_c , but $SU(4)$ violation is of order $1/N_c^2$.

The $\Delta L = 1$ interaction for three flavors is

$$\begin{aligned} \frac{\mathcal{A}^{j=1}}{N_c^2} = & \sum_{r=0}^{N_c-1} d_{1,r}^i \left(\frac{\hat{S}_1^i + \hat{S}_2^i}{N_c} \right) \left(\frac{\hat{\Lambda}_1 \cdot \hat{\Lambda}_2}{N_c} \right)^r \\ & + \sum_{r=0}^{N_c-2} d_{2,r}^i \left(\frac{\hat{\mathcal{G}}_2^{iA} \hat{T}_1^A + \hat{\mathcal{G}}_1^{iA} \hat{T}_2^A}{N_c^2} \right) \left(\frac{\hat{\Lambda}_1 \cdot \hat{\Lambda}_2}{N_c^2} \right)^r \\ & + \sum_{r=0}^{N_c-3} d_{3,r}^i \left(\frac{\hat{\mathcal{G}}_1^{iA} \hat{\mathcal{G}}_2^{jA} \hat{S}_1^j + \hat{\mathcal{G}}_2^{iA} \hat{\mathcal{G}}_1^{jA} \hat{S}_2^j}{N_c^3} \right) \left(\frac{\hat{\Lambda}_1 \cdot \hat{\Lambda}_2}{N_c^2} \right)^r \\ & + \sum_{r=0}^{N_c-1} d_{4,r}^i \epsilon \left(\frac{\hat{S}_{1s}^i + \hat{S}_{2s}^i}{N_c} \right) \left(\frac{\hat{\Lambda}_1 \cdot \hat{\Lambda}_2}{N_c^2} \right)^r \\ & + \sum_{r=0}^{N_c-3} d_{5,r}^i \epsilon \left(\frac{\hat{\mathcal{G}}_1^{iA} \hat{\mathcal{G}}_2^{jA} \hat{S}_{1s}^j + \hat{\mathcal{G}}_2^{iA} \hat{\mathcal{G}}_1^{jA} \hat{S}_{2s}^j}{N_c^3} \right) \left(\frac{\hat{\Lambda}_1 \cdot \hat{\Lambda}_2}{N_c^2} \right)^r, \end{aligned} \quad (\text{A.4})$$

which is the three-flavor generalization of (3.9). Time reversal and parity invariance requires the coefficients in (3.9) to be of the form

$$\mathbf{p}_{\text{in}} \times \mathbf{p}_{\text{out}},$$

times an arbitrary function of q^2 , k^2 and $(q \cdot k)^2$. As for the case of two flavors, the coefficients in eq. (A.4) are of order $1/N_c$, so that the $\Delta L = 1$ amplitude is of order $1/N_c^2$ relative to the central potential.

The $\Delta L = 2$ amplitude is

$$\frac{\mathcal{A}_0^{j=2}}{N_c} = \sum_{r=0}^{N_c-2} b_r^{ij} \frac{\hat{\mathcal{G}}_1^{iA} \hat{\mathcal{G}}_2^{jA}}{N_c^2} \left(\frac{\hat{\Lambda}_1 \cdot \hat{\Lambda}_2}{N_c^2} \right)^r \quad (\text{A.5})$$

where the coefficient f_r^{ij} is a symmetric traceless tensor that depends on \mathbf{p}_{in} and \mathbf{p}_{out} . This is the three-flavor generalization of (3.11).